

REES ALGEBRAS OF MODULES AND COHERENT FUNCTORS

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ABSTRACT. We show that several properties of the theory of Rees algebras of modules become more transparent using the category of coherent functors rather than working directly with modules. In particular, we show that the Rees algebra is induced by a canonical map of coherent functors.

INTRODUCTION

In [EHU03], the authors defines the Rees algebra of any finitely generated module over a noetherian ring. This definition was also studied in [Stå14], where we showed that the Rees algebra $\mathcal{R}(M)$ of a finitely generated module M is equal to the image of a canonical map $\mathrm{Sym}(M) \rightarrow \Gamma(M^*)^\vee$ from the symmetric algebra of M to the graded dual of the algebra of divided powers of the dual of the module M . In this paper, we use coherent functors to obtain nice characterizations of properties of the Rees algebra of M that are not available in the category of modules. Two of the most important of these are stated below.

For any finitely generated module M , we consider the functors $t_M = M \otimes_A (-)$ and $h^M = \mathrm{Hom}_A(M, -)$. There is a canonical map $t_M \rightarrow h^{M*}$, and we introduce the functor

$$\mathcal{G}_M = \mathrm{im}(t_M \rightarrow h^{M*}).$$

Theorem A. *Let A be a noetherian ring and let $M \rightarrow N$ be a homomorphism of finitely generated A -modules. If the induced morphism $\mathcal{G}_M \rightarrow \mathcal{G}_N$ is injective (resp. surjective), then $\mathcal{R}(M) \rightarrow \mathcal{R}(N)$ is injective (resp. surjective).*

In particular, letting $N = F$ be free, a homomorphism $M \rightarrow F$ such that the induced morphism $\mathcal{G}_M \rightarrow \mathcal{G}_F$ is injective, is a versal map in the terminology of [EHU03]. Given such a map, the theorem implies that $\mathcal{R}(M) \rightarrow \mathcal{R}(F) = \mathrm{Sym}(F)$ is injective, recovering another result of [EHU03], namely that the Rees algebra of M can be computed as the image of the map $\mathrm{Sym}(M) \rightarrow \mathrm{Sym}(F)$.

Another result we obtain, that is not available in the category of modules, is the following.

Theorem B. *There is a functor Φ from the category of coherent functors to the category of finitely generated and graded algebras over A such that $\Phi(t_M) = \mathrm{Sym}(M)$ and $\Phi(h^M) = \mathrm{im}(\mathrm{Sym}(M^*) \rightarrow \Gamma(M)^\vee)$.*

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By the result of [Stå14], this theorem shows that the Rees algebra of a module M is equal to the image of the map given by applying Φ to the canonical map $t_M \rightarrow h^{M^*}$, that is,

$$\mathcal{R}(M) = \text{im}\left(\Phi(t_M) \rightarrow \Phi(h^{M^*})\right).$$

Background. The Rees algebra of an ideal is a fundamental object in algebraic geometry. A natural question is if this object can be generalized. In [EHU03], this was done by defining the Rees algebra for any finitely generated module over a noetherian ring. Furthermore, in [Stå14], we studied this object and found an intrinsic definition in terms of the algebra of divided powers.

Coherent functors were introduced by Auslander in [Aus66], and was studied by Hartshorne in [Har98]. They are useful for various topics. In [Sch68], Schlessinger used them for describing infinitesimal deformation theory and, in [Hal12], Hall gives a nice description of Artin's criteria for algebraicity of a stack.

Structure of the article. We start in Section 1 by reviewing some results of Rees algebras of modules. In particular, we state some results about versal homomorphisms.

In Section 2 we discuss the torsionless quotient of a module M , which is defined as the image of the canonical map from M to its double dual. We find that any versal map will factor through this quotient and show that it is connected to the Rees algebra of M .

Sections 3 and 4 are focused on the study of relations between coherent functors, such as $h^M = \text{Hom}_A(M, -)$ and $t_M = M \otimes_A (-)$, and the Rees algebra. We give a characterization of versal maps in terms of coherent functors and find that their defining properties become more transparent in this setting. Generalizing the concept of the torsionless quotient of a module M to a torsionless functor \mathcal{G}_M , we show Theorem A and that a versal map $M \rightarrow F$ is equivalent to an injection $\mathcal{G}_M \hookrightarrow \mathcal{G}_F$.

Finally, in Section 5 we prove Theorem B by constructing a functor Φ from the category of coherent functors to the category of A -algebras, in such a way that the Rees algebra of a module M can be recovered as the image of the map $\Phi(t_M) \rightarrow \Phi(h^{M^*})$.

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1. VERSAL MAPS

A ring will here always describe a commutative ring with a unit and an algebra over a such a ring will always mean an associative commutative unital algebra. Throughout this paper, A will denote a noetherian ring.

Our main object of study is the Rees algebra of a finitely generated module over a noetherian ring that was defined by Eisenbud, Huneke and Ulrich in [EHU03].

Definition 1.1 ([EHU03], Definition 0.1). Let M be a finitely generated A -module. We define the *Rees algebra of M* as

$$\mathcal{R}(M) = \text{Sym}(M) / \cap_g L_g$$

where the intersection is taken over all homomorphisms $g: M \rightarrow E$ where E runs over all free modules and $L_g = \ker(\text{Sym}(g): \text{Sym}(M) \rightarrow \text{Sym}(E))$.

In [Stå14], we showed the following equivalent definition.

Theorem 1.2 ([Stå14]). *Let M be a finitely generated A -module. The Rees algebra $\mathcal{R}(M)$ of M , as defined in [EHU03], is equal to the image of the canonical map*

$$\text{Sym}(M) \rightarrow \Gamma(M^*)^\vee$$

where $\text{Sym}(M)$ denotes the symmetric algebra of M and $\Gamma(M^*)^\vee$ denotes the graded dual of the algebra of divided powers $\Gamma(M^*)$ of the dual of the module M .

Remark 1.3. Since the symmetric algebra is graded, the Rees algebra is also graded. Furthermore, the symmetric algebra preserves surjections and it follows that the Rees algebra does as well.

To compute the Rees algebra of a module M , the authors of [EHU03] introduced the notion of a versal homomorphism.

Definition 1.4 ([EHU03], Definition 1.2). Let M be a finitely generated A -module and let F be a finitely generated and free A -module. A homomorphism $\varphi: M \rightarrow F$ is *versal* if any homomorphism $M \rightarrow E$, where E is free, factors via φ .

We list a few results on versal maps. Proofs can be found in [EHU03] or [Stå14].

Proposition 1.5. *Let M be a finitely generated A -module, let F be a finitely generated and free A -module and let $\varphi: M \rightarrow F$ be a homomorphism.*

- (i) *If φ is versal, then $\mathcal{R}(M) = \mathcal{R}(\varphi)$, where $\mathcal{R}(\varphi) = \text{im}(\text{Sym}(\varphi): \text{Sym}(M) \rightarrow \text{Sym}(F))$.*
- (ii) *The map φ is versal if and only if $\varphi^*: F^* \rightarrow M^*$ is surjective.*
- (iii) *If φ is versal then it has a canonical factorization $M \rightarrow M^{**} \hookrightarrow F$, where $M^{**} \hookrightarrow F$ is injective.*

Remark 1.6. Note the following.

- (i) Given an ideal $I \subseteq A$, the inclusion $I \hookrightarrow A$ is *not* always versal, see, e.g., Remark 1.6 of [Stå14].
- (ii) A homomorphism $\varphi: M \rightarrow F$ that factors as $M \rightarrow M^{**} \hookrightarrow F$ is not necessarily versal, see, e.g., Remark 1.13 of [Stå14].

Another result from [EHU03] is the following.

Proposition 1.7. *Let M be a finitely generated module over A . Then, there exist a versal map $M \rightarrow F$ for some finitely generated and free module F .*

Remark 1.8. The versal map can be constructed as follows. Choose a finitely generated and free module F' that surjects onto the dual M^* . Then, the composition $M \rightarrow M^{**} \hookrightarrow (F')^*$ is versal as its dual is surjective by construction.

Two immediate consequences of the definition of the Rees algebra are the following. Here we prove these results using the results of [Stå14], but they can also be shown by using versal maps.

Proposition 1.9. *Let M and N be finitely generated A -modules and let $f: M \rightarrow N$ be a homomorphism.*

- (i) *If f is surjective, then $\mathcal{R}(M) \rightarrow \mathcal{R}(N)$ is surjective.*
- (ii) *If $f^*: N^* \rightarrow M^*$ is surjective, then $\mathcal{R}(M) \rightarrow \mathcal{R}(N)$ is injective.*

Proof. The map $\mathcal{R}(M) \rightarrow \mathcal{R}(N)$ is induced by the following commutative diagram.

$$\begin{array}{ccc}
 \text{Sym}(M) & \xrightarrow{\quad} & \text{Sym}(N) \\
 \downarrow & \searrow & \downarrow \\
 & \mathcal{R}(M) & \mathcal{R}(N) \\
 \downarrow & \swarrow & \downarrow \\
 \Gamma(M^*)^\vee & \xrightarrow{\quad} & \Gamma(N^*)^\vee
 \end{array}$$

If f is surjective, then $\text{Sym}(M) \rightarrow \text{Sym}(N)$ is surjective. The diagram then implies that $\mathcal{R}(M) \rightarrow \mathcal{R}(N)$ is surjective.

If $f^*: N^* \rightarrow M^*$ is surjective, then, as Γ preserves surjections, we get a surjection on the graded map $\Gamma(N^*) \rightarrow \Gamma(M^*)$. This graded map will be a surjection in every degree, and we get, by taking the graded dual, an injection $\Gamma(M^*)^\vee \rightarrow \Gamma(N^*)^\vee$. The diagram then induces an injection $\mathcal{R}(M) \rightarrow \mathcal{R}(N)$. \square

2. THE TORSIONLESS QUOTIENT

A module M is called torsionless if it can be embedded in some free module. This is equivalent to the canonical map $M \rightarrow M^{**}$ being injective, see, e.g., [Lam99, Section 4H].

Definition 2.1. For any module M we call $\text{im}(M \rightarrow M^{**})$ the torsionless quotient of M and denote it M^{tl} .

By definition we have that the torsionless quotient of M injects into the double dual M^{**} , and, if the canonical map $M \rightarrow M^{**}$ is injective, then $M^{tl} = M$. Thus, if M is torsionless, then $M^{tl} = M$. Furthermore, by Proposition 1.5.iii, we note that, given any versal homomorphism $M \rightarrow F$, the torsionless quotient of M is equal to $\text{im}(M \rightarrow F)$.

Lemma 2.2. *Given a versal map $M \rightarrow F$, then the induced map $M^{tl} \rightarrow F$ is also versal. In fact, the dual of M is equal to $(M^{tl})^*$.*

Proof. We have the following commutative diagram:

$$\begin{array}{ccc}
 M & \xrightarrow{\quad} & F \\
 \downarrow & \nearrow & \\
 M^{tl} & &
 \end{array}$$

Dualizing gives us the commutative diagram

$$\begin{array}{ccc}
 M^* & \xleftarrow{\quad} & F^* \\
 \uparrow & \nwarrow & \\
 (M^{tl})^* & &
 \end{array}$$

where the upper arrow is surjective, by Proposition 1.5.ii, since $M \rightarrow F$ is versal. Therefore, $(M^{tl})^* \rightarrow M^*$ is surjective. Furthermore, since $M \rightarrow M^{tl}$ is a surjection, we get that $(M^{tl})^* \rightarrow M^*$ is also an injection, and is therefore an isomorphism. Thus, since $F^* \rightarrow M^*$ is surjective and $M^* = (M^{tl})^*$, we get that $F^* \rightarrow (M^{tl})^*$ is surjective. By Proposition 1.5.ii, it follows that $M^{tl} \rightarrow F$ is versal. \square

From this result it follows that the torsionless quotient of M is, as the name suggests, torsionless, since the canonical map $M^{tl} \rightarrow (M^{tl})^{**} = M^{**}$ is injective. The torsionless quotient is also related to the Rees algebra as can be seen by the following results.

Lemma 2.3. *Let M be a finitely generated A -module. Then, the degree 1 part of the graded A -algebra $\mathcal{R}(M)$ is M^{tl} .*

Proof. The Rees algebra can be computed as the image of the graded A -algebra homomorphism $\text{Sym}(M) \rightarrow \text{Sym}(F)$, where $M \rightarrow F$ is a versal map to some finitely generated and free module F . Thus, the degree 1 part of $\mathcal{R}(M)$ is equal to the degree one part of $\text{im}(\text{Sym}(M) \rightarrow \text{Sym}(F))$, which is

$$\text{im}(\text{Sym}^1(M) \rightarrow \text{Sym}^1(F)) = \text{im}(M \rightarrow F) = M^{tl}. \quad \square$$

Lemma 2.4. *For any finitely generated A -module M we have that $\mathcal{R}(M) = \mathcal{R}(M^{tl})$.*

Proof. Let $M \rightarrow F$ be a versal map. This map then factors as $M \twoheadrightarrow M^{tl} \hookrightarrow F$. Since also $M^{tl} \hookrightarrow F$ is versal we get a commutative diagram

$$\begin{array}{ccccc} & & & \mathcal{R}(M^{tl}) & \\ & & & \downarrow & \\ \text{Sym}(M) & \twoheadrightarrow & \text{Sym}(M^{tl}) & \twoheadrightarrow & \text{Sym}(F) \\ & \searrow & \swarrow \theta & \uparrow & \\ & & & \mathcal{R}(M) & \end{array}$$

where the surjection $\theta: \text{Sym}(M^{tl}) \twoheadrightarrow \mathcal{R}(M)$ is canonically induced by the commutativity of the lower triangle. Thus,

$$\mathcal{R}(M^{tl}) = \text{im}(\text{Sym}(M^{tl}) \twoheadrightarrow \text{Sym}(F)) = \mathcal{R}(M). \quad \square$$

As noted in Remark 1.3, the Rees algebra preserves surjections, but an immediate consequence of the previous results is the following stronger statement.

Proposition 2.5. *Let M and N be finitely generated A -modules and let $M \rightarrow N$ be a homomorphism. Then, the induced map $M^{tl} \rightarrow N^{tl}$ is surjective if and only if $\mathcal{R}(M) \rightarrow \mathcal{R}(N)$ is surjective.*

Proof. Suppose first that $M^{tl} \rightarrow N^{tl}$ is surjective. Then, $\mathcal{R}(M^{tl}) \rightarrow \mathcal{R}(N^{tl})$ is surjective since the Rees algebra preserves surjections. From Lemma 2.4 we have that $\mathcal{R}(M) = \mathcal{R}(M^{tl})$ and $\mathcal{R}(N) = \mathcal{R}(N^{tl})$, so $\mathcal{R}(M) \rightarrow \mathcal{R}(N)$ is surjective.

Conversely, suppose that $\mathcal{R}(M) \rightarrow \mathcal{R}(N)$ is surjective. In particular, this map will be surjective in degree 1, and, by Lemma 2.3, this implies that $M^{tl} \rightarrow N^{tl}$ is surjective. \square

3. VERSAL MAPS AND COHERENT FUNCTORS

So far, we have been working in the abelian category \mathbf{Mod}_A of finitely generated A -modules over a noetherian ring A . Another abelian category of interest is the category \mathbf{Fun}_A of additive covariant functors $\mathcal{F}: \mathbf{Mod}_A \rightarrow \mathbf{Mod}_A$, where kernels, cokernels and images are all calculated pointwise. In \mathbf{Mod}_A the notions of monomorphisms and epimorphisms are equivalent to the notions of injections and surjections. A monomorphism of functors is a morphism that is injective at every point. We therefore call a monomorphism of functors an injection. Similarly, epimorphisms of functors are pointwise surjections, and we call an epimorphism of functors a surjection.

Example 3.1. Given a finitely generated module M , there is an additive covariant functor $h^M: \mathbf{Mod}_A \rightarrow \mathbf{Mod}_A$ defined by $N \mapsto h^M(N) = \mathrm{Hom}_A(M, N)$. Another example is the additive covariant functor $t_M: \mathbf{Mod}_A \rightarrow \mathbf{Mod}_A$ defined by $N \mapsto t_M(N) = M \otimes_A N$. \blacktriangle

The Yoneda embedding gives a contravariant left-exact embedding $\mathbf{Mod}_A \subset \mathbf{Fun}_A$, sending a finitely generated module M to the additive functor h^M . An immediate consequence of Yoneda's lemma is that the functors h^M are projective objects in \mathbf{Fun}_A . A functor $\mathcal{F} \in \mathbf{Fun}_A$ is called *coherent* if it has a projective resolution of the form $h^M \rightarrow h^N \rightarrow \mathcal{F} \rightarrow 0$, where $M, N \in \mathbf{Mod}_A$, and 0 denotes the zero functor. The category \mathcal{C} of coherent functors is a full subcategory of \mathbf{Fun}_A and has been well studied, see for instance [Aus66] and [Har98].

Example 3.2. Let M be a finitely generated A -module.

- (1) The functor h^M is coherent. Indeed, it has a presentation $0 = h^0 \rightarrow h^M \rightarrow h^M \rightarrow 0$.
- (2) The functor t_M is coherent. This follows since M admits a projective resolution $P_1 \rightarrow P_2 \rightarrow M \rightarrow 0$, where P_1 and P_2 are finitely generated projective modules. Since tensoring is right-exact we get an exact sequence $t_{P_1} \rightarrow t_{P_2} \rightarrow t_M \rightarrow 0$. For finitely generated projective modules P it holds that $t_P = h^{P^*}$, giving a projective resolution $h^{P_1^*} \rightarrow h^{P_2^*} \rightarrow t_M \rightarrow 0$. \blacktriangle

This example shows that the Yoneda embedding $\mathbf{Mod}_A \subset \mathbf{Fun}_A$ actually takes values in \mathcal{C} , that is, the functor $M \mapsto h^M$ is an embedding of \mathbf{Mod}_A into the category \mathcal{C} of coherent functors.

Theorem 3.3 ([Har98], Theorem 1.1a). *If $f: \mathcal{F}_1 \rightarrow \mathcal{F}_2$ is a morphism of coherent functors, then $\ker(f)$, $\mathrm{coker}(f)$, and $\mathrm{im}(f)$ are also coherent.*

One advantage that the category of coherent functors has over the category of finitely generated modules is that it has an *exact* dual.

Proposition 3.4 ([Har98], Proposition 4.1). *Let \mathcal{C} denote the category of coherent functors. There is a unique functor $\vee: \mathcal{C} \rightarrow \mathcal{C}$ which is exact, contravariant, and has the property that for any finitely generated module M , $\vee(h^M) = t_M$. Furthermore, $\vee \vee \cong \mathrm{id}_{\mathcal{C}}$.*

Remark 3.5. In the sequel we write $\vee(\mathcal{F}) = \mathcal{F}^\vee$ for any coherent functor \mathcal{F} . Note also that we use the same symbol for the dual of a coherent functor as we do for the graded dual of a graded A -algebra. This is to better emphasize the analogies that we will see further on.

Thus, analogously to the Yoneda embedding, we can consider the functor $t_-: \mathbf{Mod}_A \rightarrow \mathcal{C}$, defined by $M \mapsto t_M$, which gives a covariant right-exact embedding $\mathbf{Mod}_A \subset \mathcal{C}$. There is

also a functor $\text{ev}_A: \mathbf{Fun}_A \rightarrow \mathbf{Mod}_A$ defined by evaluating the functor at A , i.e., $\mathcal{F} \mapsto \mathcal{F}(A)$. Note that $M \mapsto t_M \mapsto t_M(A) = M$ is the identity, so evaluating the functor at A is a section of the embedding $M \mapsto t_M$, and in the spirit of algebraic geometry we call this taking global sections.

Let now $\varphi: M \rightarrow F$ be a versal map. This gives a morphism $t_M \rightarrow t_F$ of coherent functors. We saw in Proposition 1.5.iii that versal maps factors as $M \rightarrow M^{**} \hookrightarrow F$ and embedding this in the category of coherent functors gives a commutative diagram

$$\begin{array}{ccc} & t_{M^{**}} & \\ \uparrow & \searrow & \\ t_M & \longrightarrow & t_F \end{array}$$

where $t_{M^{**}} \rightarrow t_F$ is *not* injective in general, since tensoring is only right-exact. It turns out that there is another functor, not $t_{M^{**}}$, through which the map $t_M \rightarrow t_F$ has a canonical factorization, such that the second map is injective. Below, we show that this functor is h^{M^*} , resulting in the factorization (3.2).

Dualizing the map $t_M \rightarrow t_F$ gives a map $h^M \leftarrow h^F$. This we do in order to use the following result.

Proposition 3.6 ([Har98], Proposition 3.1). *Let \mathcal{F} be a (not necessarily coherent) functor. Then there is a natural map $\alpha: \mathcal{F}(A) \otimes_A (-) \rightarrow \mathcal{F}$. Furthermore, \mathcal{F} is right-exact if and only if α is an isomorphism.*

Remark 3.7. Given a module N , the homomorphism $\alpha_N: \mathcal{F}(A) \otimes_A N \rightarrow \mathcal{F}(N)$ is defined by sending, for all $a \in \mathcal{F}(A)$ and all $n \in N$, the element $a \otimes n$ to the element $\mathcal{F}(s_n)(a)$, where $s_n: A \rightarrow N$ is defined by $1 \mapsto n$. Moreover, as we can rewrite $\mathcal{F}(A) \otimes_A (-) = t_{\text{ev}_A(\mathcal{F})}(-)$, the proposition shows that there is a natural transformation $t_{\text{ev}_A(\mathcal{F})} \rightarrow \text{id}_{\mathcal{C}}$. Since $\text{ev}_A(t_M) = t_M(A) = M$, for any module M , there is also a natural transformation $\text{id}_{\mathbf{Mod}_A} \rightarrow \text{ev}_A(t_{(-)})$. One can in fact show that these natural transformations are units/counits of an adjunction between $t_{(-)}$ and ev_A .

Applying the proposition to the functor h^N , for any finitely generated module N , gives a morphism $h^N(A) \otimes_A (-) \rightarrow h^N$. Noting that $h^N(A) \otimes_A (-) = t_{N^*}$, we rewrite this as $t_{N^*} \rightarrow h^N$. By Proposition 1.5.ii, a map $M \rightarrow F$ is versal if and only if $F^* \rightarrow M^*$ is surjective. Since tensoring is right-exact, a versal map gives a surjection $t_{F^*} \twoheadrightarrow t_{M^*}$. That F is free of finite rank gives an isomorphism $t_{F^*} = h^F$. Thus, Proposition 3.6 shows that a versal map $M \rightarrow F$ induces a commutative diagram

$$(3.1) \quad \begin{array}{ccc} t_{M^*} & \longleftarrow & t_{F^*} \\ \downarrow & \nwarrow & \parallel \\ h^M & \longleftarrow & h^F \end{array}$$

which we can dualize to get the commutative diagram:

$$(3.2) \quad \begin{array}{ccc} & h^{M^*} & \\ \uparrow & \searrow & \\ t_M & \longrightarrow & t_F \end{array}$$

Taking global sections gives that $h^{M^*}(A) = M^{**}$, recovering the original factorization of Proposition 1.5.iii. However, here we see that the fact that versal maps factor via the double dual is just a special case of taking global sections of a functor $h^{M^*} = (t_{M^*})^\vee$ with the *two different duals* $*$ and \vee . Using this we are able to state a generalization of Proposition 1.5.ii.

Theorem 3.8. *Let A be a noetherian ring, let M be a finitely generated A -module and let F be a finitely generated and free A -module. Given a homomorphism $\varphi: M \rightarrow F$, the following are equivalent:*

- (i) φ is versal.
- (ii) $\varphi^*: F^* \rightarrow M^*$ is surjective.
- (iii) $\text{im}(h^F \rightarrow h^M) = \text{im}(t_{M^*} \rightarrow h^M)$.
- (iv) $\ker(t_M \rightarrow t_F) = \ker(t_M \rightarrow h^{M^*})$.
- (v) $h^{M^*} \rightarrow t_F$ is injective.

Proof. We prove this by showing the following equivalences: $(i) \Leftrightarrow (ii)$, $(ii) \Leftrightarrow (iii)$, $(iii) \Leftrightarrow (iv)$, and $(ii) \Leftrightarrow (v)$.

$(i) \Leftrightarrow (ii)$: This is Proposition 1.5.ii.

$(ii) \Leftrightarrow (iii)$: Suppose $F^* \rightarrow M^*$ is surjective. This gives the factorization (3.1), from which it follows that $\text{im}(h^F \rightarrow h^M) = \text{im}(t_{M^*} \rightarrow h^M)$.

Conversely, suppose that $\text{im}(h^F \rightarrow h^M) = \text{im}(t_{M^*} \rightarrow h^M)$. Taking global sections, we get

$$\text{im}(F^* \rightarrow M^*) = \text{im}(M^* \rightarrow M^*) = M^*,$$

that is, $F^* \rightarrow M^*$ is surjective.

$(iii) \Leftrightarrow (iv)$: These are dual statements of each other by Proposition 3.4.

$(ii) \Leftrightarrow (v)$: That $F^* \rightarrow M^*$ is surjective is equivalent to $t_{F^*} \rightarrow t_{M^*}$ being surjective. Dualizing gives that the map $h^{M^*} \rightarrow t_F$ is injective. \square

Remark 3.9. By Theorem 3.8, a homomorphism $M \rightarrow F$ is versal if and only if the induced map $h^{M^*} \rightarrow t_F$ is injective. This is a result that does *not* have an analogous statement in the category of finitely generated A -modules. Indeed, taking global sections of (v) gives an injection $M^{**} \hookrightarrow F$, but, as we saw in Remark 1.6, an A -module homomorphism $M \rightarrow F$ need not be versal even though it factors as $M \rightarrow M^{**} \hookrightarrow F$. However, this theorem shows that a versal map $M \rightarrow F$ is precisely a map that induces a factorization $t_M \rightarrow h^{M^*} \hookrightarrow t_F$.

4. TORSIONLESS FUNCTORS

Analogously to the definition of the torsionless quotient of a module from Section 2, we will now consider the torsionless quotient in the category of coherent functors.

Definition 4.1. Given a finitely generated module M we define the *torsionless quotient functor of M* as the image of the canonical map $t_M \rightarrow h^{M^*}$ and denote it by \mathcal{G}_M , that is,

$$\mathcal{G}_M = \text{im}(t_M \rightarrow h^{M^*}).$$

We call a functor $\mathcal{F} \in \mathcal{C}$ *torsionless* if $\mathcal{F} = \mathcal{G}_M$ for some finitely generated module M .

Remark 4.2. By Theorem 3.3, it is clear that \mathcal{G}_M is a coherent functor for any finitely generated module M . Furthermore, we note that $\mathcal{G}_M(A) = \text{im}(M \rightarrow M^{**}) = M^{tl}$. However, in general, \mathcal{G}_M is neither given by $t_{M^{tl}}$ nor by the image of the map $t_M \rightarrow t_{M^{**}}$. By dualizing we have that $\mathcal{G}_M^\vee = \text{im}(t_{M^*} \rightarrow h^M)$, and, in particular,

$$\mathcal{G}_M^\vee(A) = \text{im}(M^* \rightarrow M^*) = M^*.$$

Also, as the functor t_M preserves surjections it follows that \mathcal{G}_M does as well.

Lemma 4.3. *If M is a finitely generated module, then $\mathcal{G}_M = \mathcal{G}_{M^{tl}}$.*

Proof. Since $t_M \rightarrow t_{M^{tl}}$ is surjective and $M^* = (M^{tl})^*$ the result follows by the factorization $t_M \twoheadrightarrow t_{M^{tl}} \rightarrow h^{M^*}$. \square

Lemma 4.4. *If F is a free and finitely generated module, then $\mathcal{G}_F = t_F$.*

Proof. If F is free and finitely generated, then h^{F^*} is right-exact. Thus, by Proposition 3.6, $t_F \rightarrow h^{F^*}$ is an isomorphism, and the result follows. \square

Lemma 4.5. *The map $M \mapsto \mathcal{G}_M$ naturally extends to a functor $\mathcal{G}: \mathbf{Mod}_A \rightarrow \mathcal{C}$.*

Proof. Let $f: M \rightarrow N$ be a module homomorphism. This gives a commutative diagram

$$\begin{array}{ccc} t_M & \xrightarrow{\quad} & t_N \\ & \searrow & \searrow \\ & \mathcal{G}_M & \mathcal{G}_N \\ & \swarrow & \swarrow \\ h^{M^*} & \xrightarrow{\quad} & h^{N^*} \end{array}$$

which induces a morphism $\mathcal{G}_f: \mathcal{G}_M \rightarrow \mathcal{G}_N$. The other defining properties of a functor follow by similar arguments. \square

Remark 4.6. A morphism $\mathcal{G}_M \rightarrow \mathcal{G}_N$ induces a homomorphism $M^{tl} \rightarrow N^{tl}$ by taking global sections. Moreover, a morphism $\mathcal{G}_M \rightarrow \mathcal{G}_N$ is uniquely determined by the map $M^{tl} \rightarrow N^{tl}$ as the following result shows.

Proposition 4.7. *Let M and N be finitely generated modules over A . Then, a morphism $\Lambda: \mathcal{G}_M \rightarrow \mathcal{G}_N$ is uniquely determined by the map $M^{tl} \rightarrow N^{tl}$ obtained by evaluating at A .*

Proof. Evaluating at A gives a map $f: M^{tl} \rightarrow N^{tl}$. We need to show that $\Lambda = \mathcal{G}_f$. By the functoriality of the map from Proposition 3.6, we have a commutative diagram

$$\begin{array}{ccc} \mathcal{G}_M(A) \otimes_A P & \xrightarrow{\Lambda_A \otimes \text{id}} & \mathcal{G}_N(A) \otimes_A P \\ \downarrow & & \downarrow \\ \mathcal{G}_M(P) & \xrightarrow{\Lambda_P} & \mathcal{G}_N(P) \end{array}$$

for any finitely generated module P . As $\mathcal{G}_M(A) = M^{tl}$, $\mathcal{G}_N(A) = N^{tl}$, and $\Lambda_A = f$, this reduces to the commutative diagram

$$\begin{array}{ccc} t_{M^{tl}}(P) & \xrightarrow{f \otimes \text{id}} & t_{N^{tl}}(P) \\ \downarrow & & \downarrow \\ \mathcal{G}_M(P) & \xrightarrow{\Lambda_P} & \mathcal{G}_N(P) \end{array}$$

where, again by the functoriality of the map from Proposition 3.6, the vertical arrows are surjective. Thus, $\mathcal{G}_f(P) = \Lambda_P$ for any module P . \square

By this result, we have that morphisms $\mathcal{G}_M \rightarrow \mathcal{G}_N$, of torsionless functors, is equivalent to homomorphisms $M^{tl} \rightarrow N^{tl}$, of torsionless modules. From Lemma 4.3, it now follows that the full subcategory of the category of coherent functors \mathcal{C} , consisting of the torsionless functors, is equivalent to the full subcategory of the category of finitely generated modules \mathbf{Mod}_A , consisting of the torsionless modules.

Even though these categories are equivalent, we end this section by showing that there is an advantage in considering the category of torsionless functors, rather than the category of torsionless modules. In particular, Corollary 4.10 gives connections between torsionless functors and Rees algebras that do not exist between torsionless modules and Rees algebras.

Proposition 4.8. *Let $f: M \rightarrow N$ be a homomorphism of finitely generated A -modules.*

- (i) *The map $\mathcal{G}_f: \mathcal{G}_M \rightarrow \mathcal{G}_N$ is injective if and only if $f^*: N^* \rightarrow M^*$ is surjective.*
- (ii) *The map $\mathcal{G}_f: \mathcal{G}_M \rightarrow \mathcal{G}_N$ is surjective if and only if the induced map $M^{tl} \rightarrow N^{tl}$ is surjective.*

Proof. The map $t_M \rightarrow t_{M^{tl}}$ is surjective, making the diagram

$$(4.1) \quad \begin{array}{ccccc} t_{M^{tl}} & \twoheadrightarrow & \mathcal{G}_M & \hookrightarrow & h^{M^*} \\ \downarrow & & \downarrow & & \downarrow \\ t_{N^{tl}} & \twoheadrightarrow & \mathcal{G}_N & \hookrightarrow & h^{N^*} \end{array}$$

commute.

- (i) If $\mathcal{G}_M \rightarrow \mathcal{G}_N$ is injective then, dually, $\mathcal{G}_N^\vee \rightarrow \mathcal{G}_M^\vee$ is surjective. Taking global sections, we get that $N^* \rightarrow M^*$ is surjective. Conversely, if $N^* \rightarrow M^*$ is surjective, then $h^{M^*} \rightarrow h^{N^*}$ is injective, so the commutativity of diagram (4.1) implies that $\mathcal{G}_M \rightarrow \mathcal{G}_N$ is injective.
- (ii) Suppose that $\mathcal{G}_M \rightarrow \mathcal{G}_N$ is surjective. Then, taking global sections gives a surjection $M^{tl} \rightarrow N^{tl}$. Conversely, if $M^{tl} \rightarrow N^{tl}$ is surjective, then $t_{M^{tl}} \rightarrow t_{N^{tl}}$ is surjective, so (4.1) implies that $\mathcal{G}_M \rightarrow \mathcal{G}_N$ is surjective. \square

An immediate consequence of combining the previous result with Proposition 1.5.ii is the following.

Corollary 4.9. *Let M be a finitely generated A -module and let F be a finitely generated and free A -module. Then, a homomorphism $M \rightarrow F$ is versal if and only if $\mathcal{G}_M \rightarrow \mathcal{G}_F$ is injective.*

Similarly, combining Proposition 4.8 with Proposition 1.9 and Proposition 2.5 gives:

Corollary 4.10. *Let $M \rightarrow N$ be a homomorphism of finitely generated A -modules.*

- (i) *If the induced map $\mathcal{G}_M \rightarrow \mathcal{G}_N$ is injective, then the induced map $\mathcal{R}(M) \rightarrow \mathcal{R}(N)$ is injective.*
- (ii) *The map $\mathcal{G}_M \rightarrow \mathcal{G}_N$ is surjective if and only if $\mathcal{R}(M) \rightarrow \mathcal{R}(N)$ is surjective.*

5. A FUNCTOR FROM COHERENT FUNCTORS TO COMMUTATIVE ALGEBRAS

In this section, we construct a functor $\Phi: \mathcal{C} \rightarrow \mathbf{Alg}_A$ from the category of coherent functors to the category of finitely generated and graded A -algebras. This we do by first defining $\Phi(h^M)$ for any finitely generated module M . Then, for any coherent functor \mathcal{F} , we fix a presentation $h^N \xrightarrow{r} h^M \rightarrow \mathcal{F} \rightarrow 0$, and let $\Phi(\mathcal{F})$ be the coequalizer $\text{coeq}(\Phi(r), \Phi(0))$. Finally, we show that this is independent of the choice of projective resolution.

By Theorem 1.2, the Rees algebra of M is equal to the image of the canonical map $\text{Sym}(M) \rightarrow \Gamma(M^*)^\vee$. Here $\Gamma(M^*)^\vee$ denotes the graded dual of the algebra of divided powers of the dual of the module M . For definitions and properties of the algebra of divided powers and its dual, we refer the reader to [Rob63], [Ryd08] or [Stå14]. From the latter, we have the following result.

Theorem 5.1 ([Stå14], Theorem 3.9). *Let M be a finitely generated A -module. Then, the canonical module homomorphism $M^* \rightarrow \Gamma(M)^\vee$, sending M^* into the degree 1 part of $\Gamma(M)^\vee$, which is M^* , induces a natural homomorphism of graded A -algebras*

$$\text{Sym}(M^*) \rightarrow \Gamma(M)^\vee.$$

If M is free, then this map is an isomorphism.

By Remark 3.10 of [Stå14], we also note that $\Gamma(M^*)^\vee$ is in general not generated in degree 1 and is therefore hard to work with. Instead, we will consider the A -algebra

$$\mathcal{Q}(M) = \text{im}(\text{Sym}(M^*) \rightarrow \Gamma(M)^\vee),$$

which is the largest subring of $\Gamma(M)^\vee$ that is generated in degree 1.

Remark 5.2. Let M be a finitely generated A -module that is reflexive. Then, by Theorem 1.2,

$$\mathcal{Q}(M^*) = \text{im}(\text{Sym}(M^{**}) \rightarrow \Gamma(M^*)^\vee) = \text{im}(\text{Sym}(M) \rightarrow \Gamma(M^*)^\vee) = \mathcal{R}(M).$$

Thus, \mathcal{Q} is closely related to the Rees algebra. In general, we have the following result.

Lemma 5.3. *The Rees algebra $\mathcal{R}(M)$ of a finitely generated A -module M is equal to $\text{im}(\text{Sym}(M) \rightarrow \mathcal{Q}(M^*))$.*

Proof. From Theorem 1.2, we have that $\mathcal{R}(M) = \text{im}(\text{Sym}(M) \rightarrow \Gamma(M^*)^\vee)$. By the universal property of the symmetric algebra, there is a factorization

$$\text{Sym}(M) \rightarrow \text{Sym}(M^{**}) \rightarrow \Gamma(M^*)^\vee.$$

Therefore, the image of $\text{Sym}(M) \rightarrow \Gamma(M^*)^\vee$ lies within the image of $\text{Sym}(M^{**}) \rightarrow \Gamma(M^*)^\vee$. Thus,

$$\mathcal{R}(M) = \text{im}(\text{Sym}(M) \rightarrow \Gamma(M^*)^\vee) = \text{im}(\text{Sym}(M) \rightarrow \mathcal{Q}(M^*)). \quad \square$$

Theorem 5.4. *There is a functor $\Phi: \mathcal{C} \rightarrow \mathbf{Alg}_A$ such that $h^M \mapsto \mathcal{Q}(M)$.*

To give some structure to the proof, we break it down to a few lemmas. Given any finitely generated module M , we let $\Phi(h^M) = \mathcal{Q}(M)$. For any coherent functor \mathcal{F} , we now fix a projective resolution

$$h^N \xrightarrow{r} h^M \rightarrow \mathcal{F} \rightarrow 0,$$

so that $\mathcal{F} = \text{coker}(r) = \text{coeq}(r, 0)$. Then, we define $\Phi(\mathcal{F}) = \text{coeq}(\Phi(r), \Phi(0))$. That this makes Φ into a well defined functor is proved by the following results.

Lemma 5.5. *Let $r: h^N \rightarrow h^M$ be a map of coherent functors, and let $p_1: h^{M \oplus N} \rightarrow h^M$ and $p_2: h^{M \oplus N} \rightarrow h^N$ denote the projections. Then, $\text{coeq}(\Phi(r), \Phi(0)) = \text{coeq}(\Phi(\pi_1), \Phi(\pi_2))$, where $\pi_1, \pi_2: h^{M \oplus N} \rightarrow h^M$ are defined by $\pi_1 = p_1 + r \circ p_2$ and $\pi_2 = p_1$.*

Proof. The coequalizer of the two maps $\Phi(r), \Phi(0): \mathcal{Q}(N) \rightarrow \mathcal{Q}(M)$ is equal to $\mathcal{Q}(M)/I$ where I is the ideal generated by elements of the form $\Phi(r)(x) - \Phi(0)(x)$ for all $x \in \mathcal{Q}(N)$. As $\mathcal{Q}(N)$ is generated in degree 1, it follows that I is generated by $\Phi(r)(x)$ for all x in the degree 1 part of $\mathcal{Q}(N)$, which is N^* . Similarly, the coequalizer of $\Phi(\pi_1)$ and $\Phi(\pi_2)$ is $\mathcal{Q}(M)/J$, where J is the ideal generated by elements $\Phi(\pi_1)(x, y) - \Phi(\pi_2)(x, y)$ for all (x, y) in $M^* \oplus N^*$. As we, for every $(x, y) \in M^* \oplus N^*$, have that $\Phi(\pi_1)(x, y) - \Phi(\pi_2)(x, y) = \Phi(r)(y)$, we get that $I = J$. \square

Lemma 5.6. *Consider a map $f: \mathcal{F} \rightarrow \mathcal{G}$ of coherent functors. Then, there is a natural map $\Phi(f): \Phi(\mathcal{F}) \rightarrow \Phi(\mathcal{G})$.*

Proof. Let $h^N \rightarrow h^M \rightarrow \mathcal{F} \rightarrow 0$ and $h^L \rightarrow h^P \rightarrow \mathcal{G} \rightarrow 0$ be the fixed presentations of \mathcal{F} and \mathcal{G} . Then, a map $\mathcal{F} \rightarrow \mathcal{G}$ lifts to a map of complexes:

$$\begin{array}{ccccccc} h^N & \xrightarrow{r_1} & h^M & \longrightarrow & \mathcal{F} & \longrightarrow & 0 \\ \downarrow f_2 & & \downarrow f_1 & & \downarrow f & & \\ h^L & \xrightarrow{r_2} & h^P & \longrightarrow & \mathcal{G} & \longrightarrow & 0 \end{array}$$

Applying Φ to this diagram induces a map of coequalizers

$$\Phi(\mathcal{F}) = \text{coeq}(\Phi(r_1), \Phi(0)) \rightarrow \text{coeq}(\Phi(r_2), \Phi(0)) = \Phi(\mathcal{G}).$$

We need to show that this is independent of different choices of lifts of f . Given another lift g_1, g_2 of f , we get a homotopy $l: h^M \rightarrow h^L$ such that $r_2 \circ l = f_1 - g_1$. Now, letting $p_1: h^{P \oplus L} \rightarrow h^P$ and $p_2: h^{P \oplus L} \rightarrow h^L$ denote the projections, we define $\pi_1 = p_1 + r_2 \circ p_2$ and $\pi_2 = p_1$. Then, we consider the new diagram:

$$\begin{array}{ccccccc} h^N & \xrightarrow{r_1} & h^M & \longrightarrow & \mathcal{F} & \longrightarrow & 0 \\ \downarrow \parallel & & \downarrow \parallel & & \downarrow f & & \\ h^{P \oplus L} & \xrightarrow[\pi_2]{\pi_1} & h^P & \xrightarrow{p} & \mathcal{G} & \longrightarrow & 0 \end{array}$$

By Lemma 5.5, this does not alter the coequalizers of the sequences we get after applying Φ . Letting $j_1: h^P \rightarrow h^{P \oplus L}$ and $j_2: h^L \rightarrow h^{P \oplus L}$ denote the natural inclusions, we have that $\pi_1 \circ (j_1 \circ g_1 + j_2 \circ l) = f_1$ and $\pi_2 \circ (j_1 \circ g_1 + j_2 \circ l) = g_1$. Thus,

$$\Phi(f_1) = \Phi(\pi_1) \circ \Phi(j_1 \circ g_1 + j_2 \circ l),$$

and

$$\Phi(g_1) = \Phi(\pi_2) \circ \Phi(j_1 \circ g_1 + j_2 \circ l).$$

Hence, for any $x \in \mathcal{Q}(M)$, we see that the element $y = \Phi(j_1 \circ g_1 + j_2 \circ l)(x) \in \mathcal{Q}(P \oplus L)$ has the property that $\Phi(\pi_1)(y) = \Phi(f_1)(x)$ and $\Phi(\pi_2)(y) = \Phi(g_1)(x)$. Therefore, we have that $\Phi(p) \circ \Phi(f_1) = \Phi(p) \circ \Phi(g_1)$, implying that the induced maps of coequalizers are equal. \square

Proof of Theorem 5.4. Given any finitely generated module M , we let $\Phi(h^M) = \mathcal{Q}(M)$. For any coherent functor \mathcal{F} , we now fix a projective resolution

$$h^N \xrightarrow{r} h^M \rightarrow \mathcal{F} \rightarrow 0,$$

so that $\mathcal{F} = \text{coker}(r) = \text{coeq}(r, 0)$. Then, we define $\Phi(\mathcal{F}) = \text{coeq}(\Phi(r), \Phi(0))$. It remains to prove that this is independent of the choice of projective resolution.

To show this, we take another projective resolution $h^L \rightarrow h^P \rightarrow \mathcal{F} \rightarrow 0$ of \mathcal{F} . Then, we can lift the identity map $\mathcal{F} \rightarrow \mathcal{F}$ to a map of complexes:

$$\begin{array}{ccccccc} h^N & \xrightarrow{r} & h^M & \longrightarrow & \mathcal{F} & \longrightarrow & 0 \\ \downarrow & & \downarrow f & & \parallel & & \\ h^L & \xrightarrow{r'} & h^P & \longrightarrow & \mathcal{F} & \longrightarrow & 0 \\ \downarrow & & \downarrow g & & \parallel & & \\ h^N & \xrightarrow{r} & h^M & \longrightarrow & \mathcal{F} & \longrightarrow & 0 \end{array}$$

Thus, we have two lifts $\text{id}, g \circ f: h^M \rightarrow h^M$. By Lemma 5.6, it follows that these two maps induce the same map $\text{coeq}(\Phi(r), \Phi(0)) \rightarrow \text{coeq}(\Phi(r), \Phi(0))$, and this induced map must be the identity. Hence, the definition of Φ is independent of the choice of projective resolution. That Φ is well defined on morphisms now follows from Lemma 5.6. \square

Remark 5.7. The functor Φ is not right-exact. It is easy to see that Φ preserves coequalizers, but it will, in general, not preserve finite coproducts. Indeed, in Example 1.15 of [Stå14], there is a module M such that $\mathcal{R}(M \oplus M) \neq \mathcal{R}(M) \otimes_A \mathcal{R}(M)$. As that module is reflexive, we have, by Remark 5.2, that $\mathcal{R}(M) = \mathcal{Q}(M^*)$, showing that \mathcal{Q} does not preserve coproducts.

Remark 5.8. There is a reason for defining $\Phi(h^M)$ as $\mathcal{Q}(M)$, and not as $\Gamma(M)^\vee$. Indeed, if we define $\Psi(h^M) = \Gamma(M)^\vee$ and consider a presentation $h^Q \xrightarrow{r} h^P \rightarrow h^M \rightarrow 0$, then it is *not* true that $\Gamma(M)^\vee = \text{coeq}(\Psi(r), \Psi(0))$. That is because Lemma 5.5 fails for $\mathcal{Q}(M)$ replaced with $\Gamma(M)^\vee$.

Proposition 5.9. *Let M be a finitely generated A -module. Then, $\Phi(t_M) = \text{Sym}(M)$.*

Proof. Choosing a projective resolution $F_2 \rightarrow F_1 \rightarrow M \rightarrow 0$ of M gives a right-exact sequence $t_{F_2} \rightarrow t_{F_1} \rightarrow t_M \rightarrow 0$. For projective modules it holds that $t_{F_i} = h^{F_i^*}$ so we get that

$$(5.1) \quad h^{F_2^*} \xrightarrow{r} h^{F_1^*} \rightarrow t_M \rightarrow 0.$$

By Theorem 5.1, since F_1, F_2 are free, it follows that we have an isomorphism of A -algebras between $\Gamma(F_i^*)^\vee$ and $\text{Sym}(F_i)$, so $\mathcal{Q}(F_i^*) = \text{Sym}(F_i)$. Thus, applying Φ to (5.1) gives the

sequence

$$\mathrm{Sym}(F_2^*) \xrightleftharpoons[\Phi(0)]{\Phi(r)} \mathrm{Sym}(F_1^*) \longrightarrow \Phi(t_M) = \mathrm{coeq}(\Phi(\pi), \Phi(0)) .$$

As Sym preserves coequalizers it follows that $\Phi(t_M) = \mathrm{Sym}(M)$. \square

Therefore, we have, for any versal map $M \rightarrow F$, the following commutative diagram.

$$\begin{array}{ccccc} M & \xrightarrow{\quad} & M^{**} & \hookrightarrow & F \\ \downarrow t_- & & & & \downarrow t_- \\ t_M & \xrightarrow{\quad} & h^{M*} & \hookrightarrow & t_F \\ \downarrow \Phi & & \downarrow \Phi & & \downarrow \Phi \\ \mathrm{Sym}(M) & \longrightarrow & \mathcal{Q}(M^*) & \hookrightarrow & \mathrm{Sym}(F) \end{array}$$

From this, and Lemma 5.3, we see that the Rees algebra of M is given by

$$\mathcal{R}(M) = \mathrm{im}(\Phi(t_M) \rightarrow \Phi(h^{M*})) .$$

There is however no functor from the category of modules with this property. Thus, the intrinsic properties of M are better reflected in the category of coherent functors than in the category of A -modules.

Throughout this paper, we have seen many connections between the Rees algebra of a module M and the torsionless quotient functor \mathcal{G}_M . First of all, they are both given as images of canonical maps,

$$\mathcal{G}_M = \mathrm{im}(t_M \rightarrow h^{M*}) \quad \text{and} \quad \mathcal{R}(M) = \mathrm{im}(\mathrm{Sym}(M) \rightarrow \Gamma(M^*)^\vee) .$$

By the results of this section, we even see that they are both induced from the canonical map $t_M \rightarrow h^{M*}$.

Also, we showed in Sections 1 and 2 that a versal map $M \rightarrow F$ factorizes as the composition

$$M \twoheadrightarrow M^{tl} \hookrightarrow M^{**} \hookrightarrow F .$$

This versal map induces a morphism $t_M \rightarrow t_F$ of coherent functors, and we showed in Sections 3 and 4 that the previous factorization is a special case of taking global sections of the factorization given by the composition

$$t_M \twoheadrightarrow \mathcal{G}_M \hookrightarrow h^{M*} \hookrightarrow t_F .$$

Finally, we have, analogously, shown that the induced map $\mathrm{Sym}(M) \rightarrow \mathrm{Sym}(F)$ factorizes as the composition

$$\mathrm{Sym}(M) \twoheadrightarrow \mathcal{R}(M) \hookrightarrow \Gamma(M^*)^\vee \hookrightarrow \mathrm{Sym}(F) .$$

Due to the similarities of \mathcal{G}_M and $\mathcal{R}(M)$, one could have hoped that Φ would map \mathcal{G}_M to $\mathcal{R}(M)$, but, unfortunately, that is not the case as the following result shows.

Proposition 5.10. *Let M be a finitely generated A -module. Then, $\Phi(\mathcal{G}_M) = \mathrm{Sym}(M^{tl})$.*

Proof. Let $\mathcal{G}_M = \text{im}(t_M \rightarrow h^{M^*})$. Now, choose a versal map $M \rightarrow F$, where F is free, and a surjection $E \rightarrow M$, where E is free. Then, $\mathcal{G}_M = \text{im}(t_E \rightarrow t_F)$. Let $L = \text{coker}(F^* \rightarrow E^*)$ so that $h^Q = \ker(t_E \rightarrow t_F)$. In particular, we have that

$$\mathcal{G}_M = \text{im}(t_E \rightarrow t_F) = \text{coker}(f: h^L \rightarrow t_E).$$

Thus, $\Phi(\mathcal{G}_M) = \text{coeq}(\Phi(f), \Phi(0))$. There is a surjection $\text{Sym}(L^*) \rightarrow \mathcal{Q}(L)$, so it follows that

$$\Phi(\mathcal{G}_M) = \text{coeq}(\Phi(f), \Phi(0)) = \text{coeq}(\text{Sym}(L^*) \rightrightarrows \text{Sym}(E)).$$

As Sym preserves colimits, we get that

$$\Phi(\mathcal{G}_M) = \text{coeq}(\text{Sym}(L^*) \rightrightarrows \text{Sym}(E)) = \text{Sym}(\text{coeq}(L^* \rightrightarrows E)) = \text{Sym}(\text{im}(E \rightarrow F)).$$

Since $M^{tl} = \text{im}(M \rightarrow F) = \text{im}(E \rightarrow F)$ we conclude that $\Phi(\mathcal{G}_M) = \text{Sym}(M^{tl})$. \square

Remark 5.11. From this result, we see that Φ does not preserve images. It is easy to see that Φ preserves surjections, but as it does not preserve images it can not preserve injections. It could be interesting to consider derived functors of Φ to see if any new structures can be found.

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